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STABILITY ANALYSIS OF STEADY SUPERSONIC FLOW REGIMES PAST INFINITE WEDGE

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Introduction. Two solutions of the gasdynamic problem of steady supersonic flow past a wedge (Fig. 1) are known to be possible (see, for example, [1]): the *weak shock* solution (the gas flow behind the shock wave is generally supersonic, i.e., $u_0^2 + v_0^2 > c_0^2$), and the *strong shock* solution (the gas flow behind the shock wave is subsonic, i.e., $u_0^2 + v_0^2 < c_0^2$). Here u_0 and v_0 are the components of the velocity vector of the gas, and c_0 is the sound speed. Moreover, for the incident flow we have $U_\infty > c_\infty$, where c_∞ is the sound speed. An unequivocal answer to the question as to which of the two solutions actually occurs has not been found so far, in spite of the myriad papers addressing this problem. One of the possible ways of solving this question is discussed in [1] and consists in analyzing the stability of these steady gas flow regimes against small perturbations, i.e., in studying the asymptotic behavior of the solution of the linear mixed problem [see problem (1.1)–(1.4) in Sec. 1] for $t \to \infty$.

In the case when small perturbations depend on (besides the time) one "spatial" variable only, it has been rigorously shown in a number of papers (see, for example, [2, 3]) that the weak shock gas flow regime is stable against small perturbations, while the strong shock flow regime is unstable.

In the general case, it has been shown in [4] that the basic solution corresponding to supersonic flow past a wedge with a weak shock wave is stable against small perturbations, provided the gas flow behind the shock wave is supersonic and that

$$M_1(\theta) > 1 \quad \text{when} \quad \sigma \le \theta \le \theta_s. \tag{0.1}$$

Here,

$$M_1(\theta) = \frac{u_0 \cos\theta + v_0 \sin\theta}{c_0}.$$

At the same time, it has been found in [5] that the linear mixed problem [see problem (1.1)–(1.4) in Sec. 1] is also correct when

$$u_0^2 + v_0^2 < c_0^2$$

(at least for the case of small wedge angles σ; see Fig. 1). However, the stability of such flow regimes was not proved in [5].

The present paper essentially supplements the investigations undertaken in [4, 5]. In Sec. 2 we prove that the linear mixed problem (1.1)-(1.4) from Sec. 1 does not have special particular solutions that increase as $t \to \infty$ for the weak shock gas flow regime [including the case when condition (0.1) fails]. This result is also indirect confirmation of the stability of such a flow regime past at wedge for the case when condition (0.1) fails.

In Sec. 3, for the strong shock regime we construct a special particular solution that increases as $t \to \infty$, which, together with the results of [5], proves the instability of this supersonic flow regime past a wedge against perturbations.

It should also be noted that the nonexistence of a steady flow regime with a strong shock wave for tapered bodies of finite thickness has been established in several papers (see, for example, [6, 7]) by qualitative reasoning. Plausible arguments are also given in [8, 9]. In our opinion, the instability of the strong shock regime for an infinite wedge as proved in Sec. 3 and the results of [6-9] are mutually complementary.

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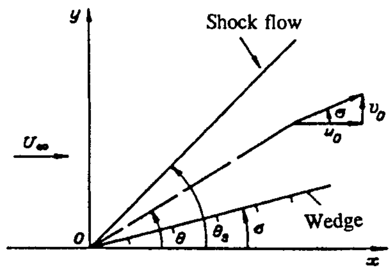

Fig. 1

1. Preliminary Information. The mathematical formulation of the problem of supersonic flow past a wedge has been given in [5]: in the region t, x > 0, and y > x tanσ the solution of the system of acoustical equations

$$\mathbf{A}\mathbf{U}_t + \mathbf{B}\mathbf{U}_x + \mathbf{C}_\sigma \mathbf{U}_y = 0, \tag{1.1}$$

is found, subject to the boundary condition at the shock wave (x = 0) and at the wedge surface (y = tanσ)

$$u_1 + du_3 = 0, \quad u_3 + u_4 = 0, \quad u_2 = \frac{\lambda}{\mu} F_y, \quad F_t + F_y \operatorname{tg}\sigma = \mu u_3; \tag{1.2}$$

$$u_2 = u_1 \operatorname{tg}\sigma \tag{1.3}$$

and the initial data at t = 0

$$\mathbf{U}(0, x, y) = \mathbf{U}_0(x, y), \qquad F(0, y) = F_0(y). \tag{1.4}$$

Here, U(t, x, y) = $(u_1\ u_2\ u_3\ u_4)^T$ is the column vector of unknown functions, and x = F(t, y) is a small displacement of the shock front, where

$$F(t, 0) = F_0(0) = 0. \tag{1.5}$$

The matrices A, B, and C_σ and the constants d, λ, and μ are given in [5]. We assume that the boundary conditions (1.2) and (1.3) conform not only with the initial data (1.4), but also with each other at points of the edge t ≥ 0 and x = y = 0. From Eqs. (1.2) and (1.3) with (1.5) we obtain

$$[\lambda + d \operatorname{tg}^2\sigma] u_3(t, 0, 0) = 0, \qquad t \geq 0,$$

i.e., if $D_1 = \lambda + d \tan^2\sigma \neq 0$, then

$$\mathbf{U}(t, 0, 0) = 0, \qquad t \geq 0. \tag{1.6}$$

Remark 1.1. The mixed problem (1.1)–(1.4) has been formulated for the case when the gas flow about a wedge with the shock wave directed along the y axis (Fig. 2) is chosen as the basic solution.

Below, we make use of an equivalent formulation, to which the mixed problem (1.1)–(1.4) (see [5]) can be reduced: In the region t, x > 0, y > x tanσ the solution of the wave equation

$$\{M^2 L_1^2 - L_2^2 - \eta^2\} u_3 = 0, \tag{1.7}$$

is found, subject to the boundary conditions at the shock wave (x = 0) and at the wedge surface (y = x tanσ)

$$\{m L_1^2 + n L_2^2 - \frac{\beta}{M^2} L_1 L_2\} u_3 = 0; \tag{1.8}$$

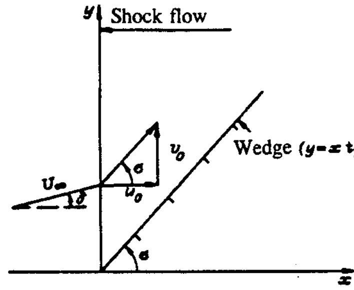

Fig. 2

$$\{\cos\sigma\eta - \sin\sigma\xi\}u_3 = 0 \tag{1.9}$$

and the initial data at t = 0. Here

$$L_1 = \frac{l_1}{\beta}; \quad l_1 = \tau + \operatorname{tg}\sigma\eta; \quad L_2 = \beta\xi - \frac{M^2}{\beta}l_1;$$

$$\tau = \frac{\partial}{\partial t}; \quad \xi = \frac{\partial}{\partial x}; \quad \eta = \frac{\partial}{\partial y};$$

$$\beta^2 = 1 - M^2\ (M^2 < 1); n = -\frac{\lambda}{\beta}; \ m = \beta d + \frac{\lambda M^2}{\beta}; \ M = \frac{u_0}{c_0} \quad \text{(see Fig. 2)}.$$

Remark 1.2. We recall once more (see also Introduction) that in [5] the correctness of the mixed problem (1.1)−(1.4) was proved when the basic solution satisfies the inequality

$$M_0 = \sqrt{\frac{u_0^2 + v_0^2}{c_0^2}} = \frac{M}{\cos\sigma} < 1$$

and the angle σ is sufficiently small, i.e., the post shock gas flow is subsonic (the basic strong shock solution. In [4] both the correctness of the mixed problem (1.1)−(1.4) and the stability of the basic solution against small perturbations were proved when the basic solution satisfies the inequalities $M_0 > 1$ and (0.1).

Introducing new independent variables

$$x' = x, \qquad y' = y - x\operatorname{tg}\sigma$$

and then omitting the primes on the variables, we rewrite problem (1.7)−(1.9):

$$\begin{aligned}
&\{M^2(\tau+\xi)^2 - (\xi - \operatorname{tg}\sigma\eta)^2 - \eta^2\}u_3 = 0, && t, x, y > 0;\\
&\{(\tau + \operatorname{tg}\sigma\eta)\big[\tau + \xi + d(\tau + \operatorname{tg}\sigma\eta) - \\
&\qquad - \frac{\xi - \operatorname{tg}\sigma\eta}{M^2}\big] + \lambda\eta^2\}u_3 = 0, && x = 0;\\
&(\eta - \sin\sigma\cos\sigma\xi)u_3 = 0, && y = 0.
\end{aligned} \tag{1.10}$$

We seek a particular solution of problem (1.10) in the form

$$u_3(t, x, y) = e^{\omega t}u(x, y), \tag{1.11}$$

where ω is a certain (in general, complex) constant. Substituting (1.11) into (1.10) we obtain the problem for the function u(x, y)

$$\{a\xi^2 + 2b\xi\eta + c\eta^2 + d_0\xi + f\}u = 0, \quad x, y > 0;$$
$$\{A_1\eta^2 - A_2\xi\eta + \omega A_3\eta - \omega \frac{\beta^2}{M^2}\xi + \omega^2(1 + d)\}u = 0, \quad x = 0; \tag{1.12}$$
$$\{\eta - \sin\sigma\cos\sigma\xi\}u = 0, \quad y = 0.$$

Here

$$a = \beta^2; \quad b = -\operatorname{tg}\sigma; \quad c = \frac{1}{\cos^2\sigma}; \quad d_0 = -2M^2\omega; \quad f = -\omega^2 M^2;$$
$$A_1 = D_1 + \frac{\operatorname{tg}^2\sigma}{M^2}; \quad A_2 = \frac{\beta^2}{M^2}\operatorname{tg}\sigma; \quad A_3 = \Big(1 + 2d + \frac{1}{M^2}\Big)\operatorname{tg}\sigma.$$

Problem (1.12) can be simplified by introducing canonical variables. Let $M_0 > 1$; then, transforming the canonical variables

$$x'' = 2\Big(y + x\frac{\operatorname{tg}\sigma}{\beta^2}\Big), \qquad y'' = 2\frac{\sqrt{\Delta}}{\beta^2}x$$

and introducing the new function v in place of the function u

$$u = \exp\Big\{\frac{M^2}{2\Delta}(\sqrt{\Delta}y'' - x''\operatorname{tg}\sigma)\omega\Big\}v(x'', y''),$$

we have the problem for v (primes are omitted)

$$v_{xx} - v_{yy} - \Omega^2 v = 0, \quad x > 0, \quad 0 < y < B_0 x;$$
$$D_1 v_{xx} - D_2 v_{xy} + D_3 v_x + D_4 v_y + D_5 v = 0, \quad y = 0; \tag{1.13}$$
$$v_y + B_0 v_x = 0, \quad y = B_0 x,$$

where

$$\Omega = \frac{\beta M}{2\Delta}\omega; \quad \Delta = M_0^2 - 1; \quad D_2 = \frac{\sqrt{\Delta}}{M^2}\operatorname{tg}\sigma; \quad D_3 = -2\frac{m\sqrt{\,}}{M}\operatorname{tg}\sigma;$$
$$D_4 = \frac{\beta\sqrt{\Delta}}{M^3}\Omega; \quad D_5 = \Big(d\frac{\beta^2}{M^2} + \lambda\frac{M^2\operatorname{tg}^2\sigma}{\beta^2}\Big)\Omega^2; \quad B_0 = \frac{\sqrt{\Delta}}{\operatorname{tg}\sigma}\,(< 1!).$$

Now let $M_0 < 1$. Transforming to the canonical variables

$$x'' = y + x\frac{\operatorname{tg}\sigma}{\beta^2}, \qquad y'' \quad -\frac{\tilde{\beta}}{\beta^2}x$$

and introducing the new function v in place of u:

$$u = \exp\Big\{\frac{M^2}{\tilde{\beta}^2}(x''\operatorname{tg}\sigma - y''\tilde{\beta})\omega\Big\}v(x'', y''),$$

we obtain from (1.12) the problem for v (primes are omitted)

$$v_{xx} + v_{yy} - \Omega^2 v = 0, \quad x > 0, \quad -B_0 x < y < 0;$$
$$D_1 v_{xx} + D_2 v_{xy} + D_3 v_x + D_4 v_y + D_5 v = 0, \quad y = 0; \tag{1.14}$$
$$v_y + B_0 v_x = 0, \quad y = -B_0 x.$$

Here

$$\Omega = \frac{\beta M}{\tilde{\beta}^2}\omega; \quad \tilde{\beta}^2 = 1 - M_0^2; \quad B_0 = \frac{\tilde{\beta}}{\operatorname{tg}\sigma}; \quad D_2 = \frac{\tilde{\beta}}{M^2}\operatorname{tg}\sigma;$$

$$D_3 = 2\frac{m\Omega}{M}\operatorname{tg}\sigma;\quad D_4 = \frac{\beta\tilde{\beta}}{M^3}\Omega;\quad D_5 = \left(d\frac{\beta^2}{M^2} + \lambda\frac{M^2\operatorname{tg}^2\sigma}{\beta^2}\right)\Omega^2.$$

Finally, let $M_0 = 1$. Assuming that

$$x'' = y + x\frac{\operatorname{tg}\sigma}{\beta^2},\qquad y'' = x$$

and introducing the new function v in place of u:

$$u = \exp\left\{\left(\frac{M^2}{\beta^2}y'' - \frac{x''}{2\operatorname{tg}\sigma}\right)\omega\right\}v(x'', y''),$$

from Eqs. (1.12) we obtain the problem for v (primes are omitted)

$$\begin{aligned} &v_{yy} + d_1 v_x = 0,\quad x > 0,\quad 0 < y < B_0 x;\\ &D_1 v_{xx} - A_2 v_{xy} + B_3 v_x + D_4 v_y + D_5 v = 0,\quad y = 0;\\ &v_y + \frac{M^2\omega}{\beta^2}v = 0,\quad y = B_0 x, \end{aligned}\tag{1.15}$$

where

$$d_1 = -2\frac{M^2\operatorname{tg}\sigma}{\beta^4}\omega;\quad B_0 = \frac{\beta^2}{\operatorname{tg}\sigma};$$

$$D_3 = -\frac{\omega}{\operatorname{tg}\sigma}(\lambda - d\operatorname{tg}^2\sigma);\quad D_4 = -\frac{\beta^2\omega}{2M^2};\quad D_5 = D_1\frac{\omega^2}{4\operatorname{tg}^2\sigma}.$$

Remark 1.3. Using the well-known oblique shock relations (see, example [10]), one readily establishes that the inequality $M_0 > 0$ can be rewritten as follows (see Fig 2):

$$\left(\operatorname{tg}^2\delta - \frac{\gamma-1}{\gamma+1}\right)M_N^4 - \frac{3-\gamma}{1+\gamma}M_N^2 + \frac{2}{\gamma+1} > 0.\tag{1.16}$$

Here $M_N = M_\infty\cos\delta$, $M_\infty = U_\infty/c_\infty$ is the Mach number of the incident flow, and $\gamma > 1$ is the adiabatic exponent. At the same time, when the expression for the coefficients d and λ for a polytropic gas (see [5]) are taken into account, the condition $D_1 \neq 0$ assumes the form

$$\left(\operatorname{tg}^2\delta - \frac{\gamma-1}{\gamma+1}\right)M_N^4 + \left(\operatorname{tg}^2\delta - \frac{3-\gamma}{1+\gamma}\right)M_N^2 + \frac{2}{\gamma+1} \neq 0.$$

Hence, for $M_0 > 1$ the condition $D_1 \neq 0$ is satisfied immediately by virtue of the inequality (1.16). In this condition the formulation of the problem (1.13) must be supplemented with the condition $v(0, 0) = 0$ (by virtue of (1.6). Furthermore, in accordance with [4] we assume that solutions of the mixed problem (1.1)−(1.4) for $M_0 > 1$ [and hence of problem (1.3)] are considered in the classes of functions that approach zero sufficiently rapidly as $x^2 + y^2 \to 0$, together with their derivatives.

2. Investigation of Problem (1.13). For simplicity we assume that the parameter in the problem (1.13) is $\Omega > 0$ (in general, $\operatorname{Re}\Omega > 0$). Utilizing the method of Reimann functions (see [11, 12]), we find the value of the function v(x, y) at the point $\tilde{M}_0(x_0, y_0)$ (Fig. 3):

$$2v(\widetilde{M}_0) = v(P) + v(Q) + \int_P^0 v[R_y + B_0 R_x]\,dx + \int_0^Q [R_y v - R v_y]\,dx,\tag{2.1}$$

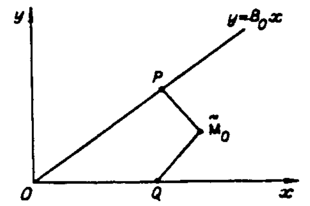

Fig. 3

($R = J_0(i\Omega K_0)$ is the Riemann function, and $K^0 = \sqrt{(x_0 - x)^2 - (y_0 - y)^2)}$. Further, since

$$R_y + B_0 R_x = i\Omega[y_0 - y - B_0(x_0 - x)]\,\frac{J_0'(i\Omega K_0)}{K_0},$$

then, superposing first the points $\tilde{M}_0$ and Q and then the points $\tilde{M}_0$ and P, from (2.1) we obtain the relations

$$\begin{aligned} f(x_0) &= g(x_0) + \int_0^{x_0} \Big\{ \frac{i\Omega B_0 x_0}{1+B_0}\,\frac{J_0'(i\Omega K)}{K}\, g(x) - J_0(i\Omega(x - x_0)) l(x) \Big\}\, dx, \\ g(x_0) &= f(Lx_0) + L \int_0^{x_0} \Big\{ \frac{i\Omega B_0 x_0}{1+B_0}\,\frac{J_0'(i\Omega\sqrt{L}K)}{\sqrt{L}K}\, f(Lx) - J_0(i\Omega\sqrt{L}K) l(Lx) \Big\}\, dx, \end{aligned} \tag{2.2}$$

where

$$f(x_0) = v(x_0, 0); \qquad g(x_0) = v\Big(\frac{x_0}{1+B_0}, \frac{B_0 x_0}{1+B_0}\Big);$$

$$l(x_0) = v_y(x_0, 0); \qquad K = \sqrt{(x_0 - x)(x_0 - Lx)}; \qquad L = \frac{1 - B_0}{1 + B_0}.$$

We add the boundary condition at $y = 0$ to the expression (2.2):

$$D_1 f''(x_0) - D_2 l'(x_0) + D_3 f'(x_0) + D_4 l(x_0) + D_5 f(x_0) = 0. \tag{2.3}$$

We further investigate expressions (2.2) and (2.3) by means of the Laplace transform (see [13]).

$$\mathcal{F}(p) = \int_0^\infty e^{-px_0} f(x_0)\, dx_0, \qquad \mathcal{L}(p) = \int_0^\infty e^{-px_0} l(x_0)\, dx_0,$$

$$G(p) = \int_0^\infty e^{-px_0} g(x_0)\, dx_0$$

be the Laplace transforms of the functions f, l, and g, and let $p = s_0 + i\sigma_0$. Applying the Laplace transform to (2.2) and (2.3), we have

$$\begin{aligned} \mathcal{F}(p) &= \frac{F_1(p)}{q_0(p)} G(q(p)) - \frac{1}{q_0(p)} \mathcal{L}(p), \\ G(p) &= \frac{I_1(p)}{\hat{q}_0(p)} \mathcal{F}\Big(\frac{\hat{q}(p)}{L}\Big) - \frac{1}{\hat{q}_0(p)} \mathcal{L}\Big(\frac{\hat{q}(p)}{L}\Big), \\ D_2 r_0 \mathcal{L}(p) &= L_1(p)\mathcal{F}(p) + L_2. \end{aligned} \tag{2.4}$$

Here,

$$q_0(p) = \sqrt{p^2 - \Omega^2}; \qquad F_1(p) = \frac{q_0(p) + B_0 p}{1 + B_0}; \qquad q(p) = \frac{p + B_0 q_0(p)}{1 + B_0};$$

$$\hat{q}_0(p) = \sqrt{p^2 - L\Omega^2}; \qquad I_1(p) = \frac{\hat{q}_0(p) + B_0 p}{1 - B_0}; \qquad \hat{q}(p) = \frac{p + B_0 \hat{q}_0(p)}{1 + B_0};$$

$$r_0(p) = p - æ_0\Omega; \qquad r_1(p) = p - \frac{\Omega}{æ_0}; \qquad æ_0 = \frac{\beta}{\mathrm{M\,tg}\,\sigma}\ (< 1);$$
$$L_1(p) = r_0^2 d\,\mathrm{tg}^2\sigma + r_1^2\lambda; \qquad L_2 = D_2 l(0) - D_1 f'(0).$$

Without loss of generality we set $L_2 = 0$ (see Remark 1.3). Then the system (2.4) can be rewritten in the form

$$\mathcal{L}(p) = \frac{L_1(p)}{D_2 r_0(p)}\mathcal{F}(p), \qquad G(p) = I_2(p)\mathcal{F}\Big(\frac{\widehat{q}(p)}{L}\Big), \qquad \qquad (2.4')$$
$$\mathcal{F}(p) = F_2(p)\mathcal{F}\Big(\frac{\widehat{q}(q(p))}{L}\Big),$$

where

$$I_2(p) = \frac{D_2 r_0\Big(\frac{\widehat{q}(p)}{L}\Big)I_1(p) - L_1\Big(\frac{\widehat{q}(p)}{L}\Big)}{D_2 r_0\Big(\frac{\widehat{q}(p)}{L}\Big)\widehat{q}_0(p)};$$
$$F_2(p) = \frac{D_2 r_0(p)F_1(p)}{D_2 r_0(p) q_0(p) + L_1(p)} I_2(q(p)).$$

Consider the third equation (2.4'). We introduce here the new variable ζ:

$$p = \frac{\Omega}{2}\Big(\zeta + \frac{1}{\zeta}\Big).$$

We then find in succession

$$q_0(p) = \frac{\Omega}{2}\Big(\zeta - \frac{1}{\zeta}\Big), \quad q(p) = \frac{\Omega}{2}\Big(\zeta + L\frac{1}{\zeta}\Big), \quad \widehat{q}_0(q(p)) = \frac{\Omega}{2}\Big(\zeta - L\frac{1}{\zeta}\Big),$$
$$F_1(p) = \frac{\Omega}{2}\Big(\zeta - L\frac{1}{\zeta}\Big), \quad \widehat{q}(q(p)) = \frac{\Omega}{2}\Big(\zeta' + \frac{1}{\zeta'}\Big), \quad \zeta' = \frac{\zeta}{L},$$
$$I_1(q(p)) = \frac{\Omega}{2}\Big(\zeta' - \frac{1}{\zeta'}\Big), \quad F_2(p) = L\frac{T(\zeta)R_2(\zeta')}{T(\zeta')R_1(\zeta)}.$$

Here

$$T(\zeta) = \zeta^2 + 1 - 2æ_0\zeta; \qquad T_0(\zeta) = \zeta^2 + 1 - \frac{2\zeta}{æ_0};$$
$$R_1(\zeta) = D_2 T(\zeta)(\zeta^2 - 1) + d\,\mathrm{tg}^2\sigma T^2(\zeta) + \lambda T_0^2;$$
$$R_2(\zeta) = D_2 T(\zeta)(\zeta^2 - 1) - d\,\mathrm{tg}^2\sigma T^2(\zeta) - \lambda T_0^2.$$

Consequently, the last equation of the system (2.4') can be rewritten as

$$\mathcal{F}(\zeta) = L\frac{T(\zeta)R_2\big(\frac{\zeta}{L}\big)}{T\big(\frac{\zeta}{L}\big)R_1(\zeta)}\mathcal{F}\Big(\frac{\zeta}{L}\Big). \qquad (2.5)$$

We now show that $\mathcal{F} \equiv 0$. To this end we require the Laplace transform $\mathcal{F}(p)$ to be defined in the half-plane Re p > Ω. The mapping

$$\zeta = \frac{1}{\Omega}\big(p + \sqrt{p^2 - \Omega^2}\,\big) \qquad (2.6)$$

is then one-sheeted and takes the region Re p > Ω onto a region D′ in the ζ plane, the branch of the root being chosen so that $\sqrt{1} = 1$. Using the boundary correspondence principle one can readily establish the form of the region D′ [this will be necessary below to justify the technique used to solve Eq. (2.5)]. Substituting $p = \Omega + i\tilde{y}$ ($\tilde{y} \in R^1$) into Eq. (2.6), we find the boundary of the region D′ in parametric form

$$\xi(\tilde{y}) = \mathrm{Re}\,\zeta = \frac{1}{\Omega}\left(\Omega + \sqrt[4]{\tilde{y}^4 + 4\Omega^2\tilde{y}^2}\cos\left(\frac{1}{2}\,\mathrm{arctg}\,\frac{2\Omega}{\tilde{y}}\right)\right),$$

$$\eta(\tilde{y}) = \mathrm{Im}\,\zeta = \frac{1}{\Omega}\left(\tilde{y} + \sqrt[4]{\tilde{y}^4 + 4\Omega^2\tilde{y}^2}\sin\left(\frac{1}{2}\,\mathrm{arctg}\,\frac{2\Omega}{\tilde{y}}\right)\right).$$

One can readily verify that $\bar{\zeta}(\tilde{y}) = \zeta(-\tilde{y})$, i.e, $\partial D'$ is symmetric about the real axis of the ζ plane. As $\tilde{y}$ approaches ∞, we obtain

$$\xi \sim \frac{\tilde{y}}{\Omega}, \quad \eta \sim \frac{\tilde{y}}{\Omega}, \quad \text{i.e.,} \quad \xi \sim \eta.$$

This result indicates that $\partial D'$ has asymptotes $\eta = (1 \pm i)\xi$, $\xi \geq 0$. It is obvious, therefore, that D′ contains a subregion D″ intercepted by lines of the form $\eta = (1 \pm \alpha i)\xi$, $\xi \geq 0$, where $\alpha > 0$ is a constant, i.e., for any number $\zeta = r_0 e^{i\varphi_0}$ in D″ it follows that D″ contains the radial $l = \{\zeta : \zeta = re^{i\varphi_0}, r > r_0\}$.

Bearing all the foregoing in mind, we find a solution of Eq. (2.5) in the region D″. To simplify our reasoning, we rewrite it as

$$g(\zeta) = f(\zeta)g(\alpha\zeta), \qquad \alpha = \frac{1}{L} > 1. \tag{2.7}$$

Here, $g(\zeta) = \mathcal{F}(\zeta)$, and $f(\zeta) = L(T(\zeta)R_2(\zeta/L))/T(\zeta/L)R_1(\zeta))$. Replacing ζ by $\alpha^k\zeta$, k = 1, 2, 3,..., in (2.7) we obtain an array of functional equations valid in the region D″, since $\alpha^k\zeta \in D''$ by virtue of the property of D″ that is contains the radial l:

$$g(\alpha^k\zeta) = f(\alpha^k\zeta)g(\alpha^{k+1}\zeta). \tag{2.8}$$

Further,

$$\frac{g(\zeta)}{g(\alpha^j\zeta)} = \frac{g(\zeta)}{g(\alpha\zeta)}\frac{g(\alpha\zeta)}{g(\alpha^2\zeta)}\cdots\frac{g(\alpha^{j-1}\zeta)}{g(\alpha^j\zeta)} = \prod_{k=0}^{j-1} f(\alpha^k\zeta)$$

or

$$g(\zeta) = g(\alpha^j\zeta)\prod_{k=0}^{j-1} f(\alpha^k\zeta).$$

We recall that the function $f(\zeta)$ is known and defined in the region D′. Letting $j \to \infty$, we obtain the formal solution of Eq. (2.7)

$$g(\zeta) = g(\infty)\prod_{k=0}^{\infty} f(\alpha^k\zeta) = g(\infty)A(\zeta). \tag{2.9}$$

Note that each solution of Eq. (2.7) can be represented in the form (2.9), utilizing (2.8). Further, if $g(\zeta)$ is a solution of Eq. (2.7), the function $\gamma g(\zeta)$ ($\gamma \in C^1$) is also a solution of this equation. As a result, all solutions of Eq. (2.7) can be written in the form

$$g(\zeta) = \gamma A(\zeta). \tag{2.10}$$

We now show that $A(\zeta) \equiv 0$. It then follows from (2.10) that the only solution of Eq. (2.7) is identically zero. To this end we consider the function

$$\begin{aligned} A_n(\zeta) &= \prod_{k=0}^{n} f\left(\frac{\zeta}{L^k}\right) = L^{n+1}\frac{T(\zeta)}{T\left(\frac{\zeta}{L^{n+1}}\right)}\prod_{k=0}^{n}\frac{R_2\left(\frac{\zeta}{L^{k+1}}\right)}{R_1\left(\frac{\zeta}{L^k}\right)} = \\ &= L^{n+1}\frac{T(\zeta)}{T\left(\frac{\zeta}{L^{n+1}}\right)}\prod_{k=0}^{n}\frac{a_k}{b_k}, \end{aligned} \tag{2.11}$$

where $a_k = R_2(\zeta/L^{k+1})$, $b_k = R_1(\zeta/L^k)$, ζ is fixed, and n is a sufficiently large integer. It follows from (2.11) that

$$\begin{aligned} |A_n(\zeta)| &= \left|L^{n+1}\frac{T(\zeta)}{T\left(\frac{\zeta}{L^{n+1}}\right)}\right|\prod_{k=0}^{n}\frac{|a_k|}{|b_k|} = \left|L^{n+1}\frac{T(\zeta)}{T\left(\frac{\zeta}{L^{n+1}}\right)}\right|\prod_{k=0}^{N}\frac{|a_k|}{|b_k|}\prod_{k=N+1}^{n}\frac{|a_k|}{|b_k|} = \\ &= |\Phi_{N;n}(\zeta)|\prod_{k=N+1}^{n}\frac{|a_k|}{|b_k|} \le |\Phi_{N;n}(\zeta)|\prod_{k=N+1}^{n}\frac{|D_2 - D_1| + \varepsilon}{|D_2 + D_1| - \varepsilon} = \\ &= |\Phi_{N;n}(\zeta)|\left(\frac{|D_2 - D_1| + \varepsilon}{|D_2 + D_1| - \varepsilon}\right)^{n-N}, \quad \Phi_{N;n}(\zeta) = L^{n+1}\frac{T(\zeta)}{T\left(\frac{\zeta}{L^{n+1}}\right)}\prod_{k=0}^{N}\frac{a_k}{b_k}, \end{aligned}$$

since obviously, $a_k \to D_2 - D_1$ and $b_k \to D_2 + D_1$ in the limit $k \to \infty$, and so there exists an N such that the inequalities $|a_k - (D_2 - D_1)| < \varepsilon$ and $|b_k - (D_2 + D_1)| < \varepsilon$ hold for $k > N$, where $\varepsilon > 0$ is a sufficiently small number. Since $D_1 > 0$ [inasmuch as $M_0 > 1$; see (1.16)], then

$$\frac{|D_2 - D_1|}{|D_2 + D_1|} < 1.$$

Hence, the following inequality is satisfied for small values of ε:

$$\frac{|D_2 - D_1| + \varepsilon}{|D_2 + D_1| - \varepsilon} \le 1 - \delta < 1$$

(the value of δ is sufficiently small). We finally have

$$|A_n(\zeta)| \le |\Phi_{N;n}(\zeta)|(1 - \delta)^{n-N}$$

for fixed N.

Proceeding to the limit $n \to \infty$, we have $|A_n(\zeta)| \to 0$, i.e., $A(\zeta) = A_\infty(\zeta) \equiv 0$ in D″. Continuing the function $A(\zeta)$ analytically into the region D′, we find that $A(\zeta) \equiv 0$ in D′, so that $\mathcal{F}(\zeta) = 0$ (consequently, $\mathcal{F}(p) \equiv 0$ as well). From the system (2.4′) we then obtain $\mathcal{L}(p) = G(p) \equiv 0$. Reverting to the original functions, we obtain $f(x_0) = g(x_0) = l(x_0) \equiv 0$, and it follows immediately from (2.1) that $v \equiv 0$ as well.

3. Investigation of the Spectral Problem (1.14). Let us consider the spectral problem (1.14). Introducing polar coordinates, we obtain

$$
\begin{gathered}
v_{rr} + \frac{1}{r^2} v_{\theta\theta} + \frac{1}{r} v_r - \Omega^2 v = 0, \quad r > 0, \quad \theta \in (-\theta_0, 0); \\
D_1 v_{rr} + D_2 \Big\{ \frac{1}{r} v_{r\theta} - \frac{1}{r^2} v_\theta \Big\} + D_3 v_r + D_4 \frac{1}{r} v_\theta + D_5 v = 0, \ r > 0, \ \theta = 0; \\
v_\theta = 0, \quad r > 0, \quad \theta = -\theta_0.
\end{gathered}
\tag{3.1}
$$

Here, $\tan \theta_0 = B_0$ (B_0 is defined in Sec. 1). Note that functions of the form

$$\cos n(\theta + \theta_0) J_n(-i\Omega r), \qquad n \geq 2,$$

satisfy the first and third equations of the system (3.1); moreover, they possess the necessary asymptotic behavior as $r \to 0$ (see [5, 4]). We therefore seek a solution of problem (3.1) in the form

$$v_0(r, \theta) = \sum_{n \geq 2} A_n \cos n(\theta + \theta_0) J_n(-i\Omega r). \tag{3.2}$$

We introduce (3.2) into the boundary condition at $\theta = 0$ and try to determine the constants A_n:

$$
\begin{gathered}
\sum_{n\geq 2} D_1 A_n (-i\Omega)^2 J_n''(-i\Omega r) \cos n\theta_0 + \sum_{n \geq 2} D_2 A_n \frac{i\Omega n}{r} J_n'(-i\Omega r) \sin n\theta_0 - \\
- \sum_{n\geq 2} D_2 A_n \frac{-n}{r^2} J_n(-i\Omega r) \sin n\theta_0 + \sum_{n\geq 2} D_3 A_n (-i\Omega) J_n'(-i\Omega r) \cos n\theta_0 + \\
+ \sum_{n\geq 2} D_4 A_n \frac{-n}{r} J_n(-i\Omega r) \sin n\theta_0 + \sum_{n\geq 2} D_5 A_n J_n(-i\Omega r) \cos n\theta_0 = 0.
\end{gathered}
\tag{3.3}
$$

Making use of the known relations between Bessel functions of different orders and their derivatives (see 13]), we eliminate the derivatives of the function J_n and terms of the form $(1/r^k)J_n$ in (3.3); we finally obtain

$$
\begin{gathered}
\sum_{n\geq 2} \Big[\frac{D_1 \cos n\theta_0 - D_2 \sin n\theta_0}{4} A_n + i \frac{\widehat{D}_3 \cos(n-1)\theta_0 - \widehat{D}_4 \sin(n-1)\theta_0}{2} A_{n-1} - \\
- \frac{D_1 + 2\widehat{D}_5}{2} \cos(n-2)\theta_0 A_{n-2} - i \frac{\widehat{D}_3 \cos(n-3)\theta_0 + \widehat{D}_4 \sin(n-3)\theta_0}{2} A_{n-3} + \\
+ \frac{D_1 \cos(n-4)\theta_0 + D_2 \sin(n-4)\theta_0}{4} A_{n-4} \Big] J_{n-2}(-i\Omega r) = 0,
\end{gathered}
$$

where

$$\widehat{D}_3 = 2 \frac{m \operatorname{tg} \sigma}{M}; \qquad \widehat{D}_4 = \frac{\beta \widetilde{\beta}}{M^3}; \qquad \widehat{D}_5 = \Big(d \frac{\beta^2}{M^2} + \lambda \frac{M^2 \operatorname{tg}^2 \sigma}{\beta_\zeta^2} \Big).$$

Let

$$A_{-2} = A_{-1} = A_0 = A_1 = 0.$$

Then, assuming that

$$\frac{D_1 \cos 2\theta_0 - D_2 \sin 2\theta_0}{4} A_2 = 0,$$

i.e.,

$$\operatorname{tg} 2\theta_0 \doteq \frac{D_1}{D_2}, \qquad A_2 \in R^1 \backslash \{0\}, \tag{3.4}$$

we find the remaining coefficients recusively.

Remark 3.1. The coefficient D_1 is positive for weak shock regimes (see Remark 1.3), is negative for strong shock regimes, and is equal to zero for the regime corresponding to the maximum angle of flow deflection at an oblique shock (see [3]).

Let us verify that the constructed solution actually corresponds to the strong shock regime. To do this, it is sufficient to confirm that $D_1 < 0$ when condition (3.4) is satisfied. Writing the relation $\tan 2\theta_0 = D_1/D_2$ in the form

$$z^2[M^2(2+d(1+M^2))] + z[M^2\lambda(1+M^2) - (dM^2+2)(1-M^2)] - \lambda M^2(1-M^2) = 0, \quad z = \mathrm{tg}^2\sigma,$$

we find its roots z_+ and z_-. We substitute the smaller root z_- into the expression for D_1 and prove the inequality $D_1 < 0$. We have the relation

$$d\Big(2 + dM^2 + \lambda M^2\frac{1+M^2}{\beta^2}\Big) + \frac{4\lambda M^2}{\beta^2} < d\sqrt{\Big(2 + dM^2 - \lambda M^2\frac{1+M^2}{\beta^2}\Big)^2 + \frac{4\lambda M^4}{\beta^2}(2+d(1+M^2))}.$$

Squaring it, we obtain

$$\frac{16\lambda^2M^4}{\beta^4} + 8\lambda dM^2\Big(2 + \lambda M^2\frac{1+M^2}{\beta^2}\Big) < \frac{8\lambda d^2M^2}{\beta^2}(1+M^2)$$

or

$$\frac{8\lambda M^2}{\beta^2}\Big(d + \frac{\lambda M^2}{\beta^2}\Big)(2 + d(1+M^2)) < 0.$$

The latter inequality is correct, since the conditions $(d + \lambda M^2/\beta^2) > 0$ and $\lambda < 0$ are satisfied for a polytropic gas. Consequently, the constructed solution (formal so far) does indeed correspond to the strong shock flow regime.

Remark 3.2. The relation $\tan^2\sigma = z_-$ is also an algebraic equation for δ as a function of M_∞ (σ and δ are bound by a well-known relation (see [10] and Fig. 2)), which, for example, can be solved trivially for sufficiently large values of M_∞. Consequently, values of M_∞ and δ can be found such that the relation (3.4) is satisfied.

We now prove the convergence of the series (3.2) subject to condition (3.4). Calculating the coefficients A_n, we see that for large values of n

$$|A_n| \sim 2^{n-3}\prod_{k=3}^{n}\Big|\frac{\widehat{D}_3\cos(k-1)\theta_0 - \widehat{D}_4\sin(k-1)\theta_0}{D_1\cos k\theta_0 - D_2\sin k\theta_0}\Big| + (|D_1|+|\widehat{D}_5|)^m(\widehat{D}_3+\widehat{D}_4)^s\prod_{k=2}^{n}\frac{1}{|D_1\cos k\theta_0 - D_2\sin k\theta_0|},$$

where m and s increase with increasing n, and $m + s \le n$. We assume that $D_1 \cos k\theta_0 - D_2 \sin k\theta_0 \neq 0$, because if $D_1 \cos k\theta_0 - D_2 \sin k\theta_0 = 0$, then the coefficient of A_k is equal to zero, and this term will not appear in the series (3.3), i.e., such factors are not involved in the product. Further, it is obvious that $|D_1 \cos k\theta_0 - D_2 \sin k\theta_0| > b > 0$ for all values of k (b is a constant independent of k). Finally, $|A_n| \sim a^n$, where a is a certain number. Let r be fixed. Since

$$|v_0(r,\theta)| \le \sum_{n\ge 2}|A_n|\,|J_n(-i\Omega r)|, \qquad |J_n(-i\Omega r)| \le c\,\frac{\mathrm{ch}(\Omega r)}{(n-1)!},$$

and

$$|v_0(r,\theta)| \sim \sum_{n\geq 2} \frac{a_1^n}{(n-1)!}.$$

Thus, the series (3.2) converges pointwise. Obviously, however, the series (3.2) converges uniformly in any region of the form $D' = \{(r, \theta) : 0 \leq r \leq r^* < \infty, -\theta_0 \leq \theta \leq 0\}$. Moreover, the function $v_0(r, \theta)$ is infinitely differentiable in D'. To obtain a "solution" in the initial region $D = \{(r, \theta) : 0 \leq r < \infty, -\theta_0 \leq \theta \leq 0\}$, we set $v(r, \theta) \equiv 0$ in $D \backslash D'$.

Let us determine the eigenvalue to which the resulting "solution" corresponds. To do so, we integrate the first equation of (3.1) over the region D:

$$\int_D \Delta v \, dx\, dy = \Omega^2 \int_D v \, dx\, dy.$$

Invoking the Gauss theorem, we obtain

$$\Omega^2 \int_D v\, dx\, dy = \int_{\partial D} \frac{\partial v}{\partial n}\, dl = \int_{\partial D_1} \frac{\partial v}{\partial n}\, dl + \int_{\partial D_2} \frac{\partial v}{\partial n}\, dl,$$

where $\partial D_1 = \{(r, \theta) : r > 0, \theta = -\theta_0\}$; $\partial D_2 = \{(r, \theta) : r > 0, \theta = 0\}$. The integral $\int_{\partial D_1} \frac{\partial v}{\partial n}\, dl = 0$ is equal to zero by virtue of the impermeability condition Then

$$\Omega^2 \int_D v\, dx\, dy = \int_{\partial D_2} \frac{\partial v}{\partial n}\, dl. \tag{3.5}$$

But $\frac{\partial v}{\partial n}\big|_{\partial D_2} = \frac{\partial v}{\partial y}$. Expressing $\partial v/\partial y$ from the condition at the shock wave, we find

$$\frac{\partial v}{\partial y} = -\frac{1}{D_4}(D_1 v_{xx} + D_2 v_{xy} + D_3 v_x + D_5 v). \tag{3.6}$$

Substituting (3.6) into (3.5) and making use of the fact that

$$v,\ v_x,\ v_y \to 0 \quad \text{and} \quad r \to 0 \quad \text{as} \quad r \to \infty, \tag{3.7}$$

we have

$$\Omega = -\frac{\widehat{D}_5}{\widehat{D}_4} \frac{\int_{\partial D_2} v\big|_{\partial D_2} dl}{\int_D v\, dx\, dy}. \tag{3.8}$$

Remark 3.3. The conditions (3.7) are satisfied for all solutions of the problem (3.1) (see [14]).

Introducing the relation (3.2) into (3.8), we obtain

$$\Omega = -\frac{\widehat{D}_5}{\widehat{D}_4} \frac{\sum_{n\geq 2} A_n \cos n\theta_0 \int_0^{r^*} J_n(-i\Omega r)\, dr}{\sum_{n\geq 2} A_n (\sin n\theta_0)/n \int_0^{r^*} r J_n(-i\Omega r)\, dr}. \tag{3.9}$$

Finally, we have the algebraic equation for Ω (the value of r^* is fixed). We next investigate the solvability of Eq.(3.9). To this end, we consider the function

$$F(\Omega) = \Omega + \frac{\widehat{D}_5}{\widehat{D}_4} \frac{\sum\limits_{n\geq 2} A_n \cos n\theta_0 \int\limits_0^{r^*} J_n(-i\Omega r)\, dr}{\sum\limits_{n\geq 2} A_n(\sin n\theta_0)/n \int\limits_0^{r^*} r J_n(-i\Omega r)\, dr}.$$

Further, if A_2 is assumed to be real, then both A_n and $J_n(-i\Omega r)$ will be real for even order index and purely imaginary for odd index, i.e., F is a real function. As $\Omega \to \infty$, the fraction is bounded, hence $F(+\infty)$, and $F(0) = \frac{8}{3r^*} \frac{\widehat{D}_5}{\widehat{D}_4} \frac{D_2}{D_1} < 0$. Then, considering $F(\Omega)$ in the interval [0, B], where the value of B is sufficiently large, we infer that $F(\Omega)$ is continuous therein and at the ends assumes values of different signs for positive $\hat{D}_5$ (which, in turn, is possible for small values of σ). Furthermore, $F'_\Omega \mid_{r^*=0} > 0$, so that for sufficiently small values of r^* the function $F(\Omega)$ increases monotonously. Consequently, the equation $F(\Omega) = 0$ has a single positive root Ω_1, which can be found, say, by the bisection algorithm (see, for example [15]). We submit this root as the eigenvalue of problem (1.14). The formulas of Sec. 1 now imply that $\omega > 0$. As a result, no matter how small the initial data for problem (1.10) in the norm, we still have unbounded growth of the solution with time in any reasonable norm.

Remark 3.4. As a matter of fact, the above solution does not have any properties that would be desirable in our case, since an *a priori* estimate guaranteeing the correctness of the initial formulation of the problem has been obtained in the class of functions in space W_2^2. Hence, these functions can be adjusted on a set of measure zero so as to be continuous (see, for example, [5]). The particular "solution" obtained in this section does not possess such a property (it has a line of nonremovable discontinuity $r = r^*$, $-\theta_0 < \theta < 0$).

The situation can be corrected if we consider a problem of the form

$$\begin{gathered} v_{rr} + \frac{1}{r^2} v_{\theta\theta} + \frac{1}{r} v_r - \Omega_1^2 v = 0, \qquad r^* < r < r^* + \varepsilon, \qquad \theta \in (-\theta_0, 0); \\ D_1 v_{rr} + D_2 \left\{ \frac{1}{r} v_{r\theta} - \frac{1}{r^2} v_\theta \right\} + D'_3 v_r + D'_4 \frac{1}{r} v_\theta + D'_5 v = 0, \\ r^* < r < r^* + \varepsilon, \qquad \theta = 0; \\ v = 0, \qquad r = r^* + \varepsilon, \qquad \theta \in (-\theta_0, 0); \\ v = v_0(r^*, \theta), \qquad r = r^*, \qquad \theta \in (-\theta_0, 0). \end{gathered} \tag{3.10}$$

Here the primed coefficients differ from the unprimed in that they contain Ω_1 instead of Ω, and ε is a small number chosen so that Ω_2 will remain positive (see the discussion below).

Problem (3.10) is certainly solvable (see [14]). We denote its solution by v_1. Substituting the function

$$v = \begin{cases} v_0 & \text{for} \quad 0 < r < r^*, \qquad \theta \in (-\theta_0, 0); \\ v_1 & \text{for} \quad r^* < r < r^* + \varepsilon, \quad \theta \in (-\theta_0, 0); \\ 0 & \text{for} \quad r^* + \varepsilon < r, \qquad \theta \in (-\theta_0, 0), \end{cases} \tag{3.11}$$

into Eq. (3.8) we find a new value of Ω_2, which, in turn, is introduced into (3.2) and (3.10). After this is done, we consider the function v^1, which is analogous to (3.11), the only difference being that v^1 contains, instead of v_1, the solution of (3.10) corresponding to the number Ω_2, etc. Thus, the function v_0 can be regarded as an approximation of the particular "solution" of W_2^2 which increases exponentially with time.

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